Tikrit university
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Lectures on Engineering Analysis

Chapter 2
Gamma and Beta Functions

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Gamma and Beta functions

In the chapter we look at two special functions of improper integrals knows as Gamma and Beta. These functions play important role in applied mathematics.

Definitions

The beta function is denoted *β (m, n)* and defined by the integral

$$\beta\,(m,n) = \int_0^1 x^{\mathrm{m}-1}(1-x)^{n-1}\,dx \quad (m, n > 0)$$

Gamma function

If $n > 0$, the gamma function *Γ(n)* is defined by the improper integral. It is sometimes called the second Eulerian integral and defined by the integral

$$\Gamma\,(n) = \int_0^\infty x^{n-1}\,e^{-x}\,dx$$

is called gamma function and is denoted by ***Γ (n)***

Example *Γ (1)*

$$\Gamma\,(1) = \int_0^\infty x^{1-1}\,e^{-x}\,dx$$

$$\Gamma\,(1) = \int_0^\infty x^{0}\,e^{-x}\,dx$$

$$= \int_0^\infty x^{0}\,e^{-x}\,dx$$

$$= [\,-e^{-x}]\,_0^\infty$$

$$= \lim_{x\to\infty}(-\,e^{\infty}) - (-\,e^{0})$$

$$= 0 - (-1) = 1$$

Theorem

$n > 0 \quad \Gamma(n+1) = n\,\Gamma(n)$

Proof :

By definition of gamma function

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$$

Replacing n by (n+1) in the definition of gamma function. where $n = (n + 1)$

$$\Gamma(n+1) = \int_0^\infty x^n \; e^{-x}\, dx \qquad \textbf{(1)}$$

We can integrate by parts

$u = x^n \quad dv = e^{-x}$

$du = n\, x^{n-1} \quad v = -\, e^{-x}$

on integrating by parts, we get

$$= \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty n x^{n-1} \; (-1) e^{-x}\, dx$$

$$= n \int_0^\infty x^{n-1} \; e^{-x}\, dx$$

By substituting this result in the equation 1, we get

$\Gamma(n+1) = n\,\Gamma(n)$

If n is positive, we can derive the following

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$= n\,\Gamma(n-1+1)$$

$$= n\,(n-1)\,\Gamma(n-1) \text{ using above result}$$

$$= n\,(n-1)\,(n-2)\,\Gamma(n-2)$$

$$= \ldots\ldots = n\,(n-1)\,(n-2)\,\ldots\ldots\ldots\ldots\,(2)\,(1)\,\Gamma(1)$$

$$= n!\,\Gamma(1) \qquad \text{we know } \Gamma(1) = 1$$

So the relationship between gamma and factorial :

$$\Gamma(n+1) = n!$$

For example s $\Gamma(2) = 1! = 1$, $\Gamma(3) = 2! = 2$, $\Gamma(4) = 3! = 6$

Therefore the Factorial function:

$$= \int_0^\infty x^n \quad e^{-x}\,dx = n!$$ ***n* (=1, 2, 3, ……etc.) is a positive integer.**

For example

For $n = 0 \implies \int_0^\infty x^0 \quad e^{-x}\,dx = 1 = 0!$

For $n = 1 \implies \int_0^\infty x^1 \quad e^{-x}\,dx = 1 = 1!$

Theorem

$$\Gamma\,(1/2) = \sqrt{\pi}$$

Proof : By definition of gamma function

$$\Gamma\,(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$$

Put $x^n = y$ and $nx^{n-1}dx = dy$ we get

$$\Gamma\,(n) = \frac{1}{n} \int_0^\infty e^{-y^{1/n}}\, dy$$

Now put $\mathrm{n} = \frac{1}{2}$ in above equation, we get

$$\Gamma\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\sqrt{\pi}\right) = \sqrt{\pi}$$

Examples

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$n > 0 \quad \Gamma(n+1) = n\,\Gamma(n)$

$$\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3}{2}\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\,\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{15}{8}\Gamma\left(\frac{1}{2}\right).$$

Examples:

$$\begin{cases} \Gamma(1) = 0! \\ \Gamma(2) = 1! \\ \Gamma(3) = 2! \\ \Gamma(4) = 3! \end{cases}$$

Example

Evaluate $\int_0^\infty x^4 e^{-x} dx$

Solution

$$\int_0^\infty x^4 e^{-x} dx = \int_0^\infty x^{5-1} e^{-x} dx = \Gamma(5)$$

$$\Gamma(5) = \Gamma(4+1) = 4! = 4(3)(2)(1) = 24$$

Examples

Evaluate $\int_0^\infty x^{1/2} e^{-x} dx$

$$\int_0^\infty x^{1/2} e^{-x} dx = \int_0^\infty x^{3/2-1} e^{-x} dx = \Gamma(3/2)$$

$$3/2 = \tfrac{1}{2} + 1$$

$$\Gamma(3/2) = \Gamma(\tfrac{1}{2}+1) = \tfrac{1}{2}\,\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi}$$

Example **Evaluate $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$**

<u>Solution</u>: Let $I = \int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}}\, dx$

Putting $\sqrt{x} = t \implies x = t^2$ so that $dx = 2t\,dt$ in (i), we get

$$I = \int_0^\infty t^{1/2} e^{-t}\, 2t\, dt$$

$$= 2\int_0^\infty t^{3/2} e^{-t}\, dt$$

$$= 2\int_0^\infty t^{\frac{5}{2}-1} e^{-t}\, dt$$

$$= 2\Gamma\left(\frac{5}{2}\right)$$

$$= \left(2 \times \frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)$$

$$= \left(2 \times \frac{3}{2} \times \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{2}\sqrt{\pi}$$

$$\therefore \quad \int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx = \frac{3}{2}\sqrt{\pi}$$

Example Evaluate $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}} dx$.

Solution: Let $I = \int_0^\infty \sqrt{x} e^{-\sqrt[3]{x}} dx$ .... (1)

Putting $\sqrt[3]{x} = t$ or $x = t^3$, $dx = 3t^2 dt$ in (1) we get

$$I = \int_0^\infty t^{3/2} e^{-t}\, 3t^2 dt = 3\int_0^\infty t^{7/2} e^{-t}\, dt = 3\Gamma\left(\frac{9}{2}\right) = 3\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{315}{16}\sqrt{\pi} \quad \text{Ans.}$$

Example

Find the value of the integral $I = \int_0^\infty x^6 \; e^{-4x^2} \; dx$

soulution

$$let\ y = 4x^2 \Rightarrow x = \frac{y^{1/2}}{2} \qquad dx = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) y^{-1/2}\, dy \qquad dx = \left(\frac{1}{4}\right)\frac{dy}{\sqrt{y}}$$

$$I = \int_0^\infty x^6 \; e^{-4x^2} \; dx$$

$$I = \int_0^\infty \left(\frac{y^{1/2}}{2}\right)^6 \; e^{-y} \frac{1}{4}\frac{dy}{\sqrt{y}}$$

$$I = \frac{1}{4}\int_0^\infty \frac{y^3}{64}\; \frac{1}{y^{1/2}}\, e^{-y}\, dy$$

$$I = \frac{1}{256}\int_0^\infty y^{5/2}\, e^{-y}\, dy$$

$$I = \frac{1}{256}\int_0^\infty y^{7/2-1}\, e^{-y}\, dy \quad \Rightarrow \quad I = \frac{1}{256}\Gamma\left(\frac{7}{2}\right) \quad \Rightarrow \quad I = \frac{15\sqrt{\pi}}{2048}$$

Theorem

If $n > 0, then \dfrac{\Gamma(n)}{z^n} = \int_0^{\infty} e^{-zx}\ x^{n-1}\, dx$

Proof

We know that $\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx$

On putting $x = az$ in the above equation, we get

$$\Gamma(n) = \int_0^{\infty} e^{-az}\ (az)^{n-1}\ a\, dz$$

$$= \int_0^{\infty} e^{-az}\ z^{n-1}\ a^{n}\, a^{-1} a\, dz$$

$$= a^{n} \int_0^{\infty} e^{-az}\ z^{n-1}\, dz$$

Replacing a by z in the above equation, we get

$$= a^{n} \int_0^{\infty} e^{-ax}\ x^{n-1}\, dx$$

$$\frac{\Gamma(n)}{z^n} = \int_0^{\infty} e^{-zx}\ x^{n-1}\, dx$$

Beta function

If $m > 0, n > 0, then\ Beta$ function is defined by the integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ and is denoted by $\beta\ (m,n)$

$$i.e.\ \beta\ (m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Where $m\ and\ n$ are positive, and this integral is convergent for $m > 0\ and\ n > 0$.

$$\int_0^{\pi/2} sin^{2m-1}\theta\ cos^{2n-1}\theta\ d\theta = \frac{1}{2}\beta(m,n) = \frac{\Gamma\,(m)\,\Gamma\,(n)}{2\,\Gamma\,(m+n)}$$

Proof : By definition of beta function

$$\beta\ (m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Theorem

if $m\ and\ n$ are positive integers, then

$$\beta\ (m,n) = \frac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}$$

But applying the define integral property

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Hence, we can say that

$$\beta\,(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Now integrating by parts, we get

$$B(m,n) = \left[(1-x)^{m-1}\frac{x^n}{n}\right]_0^1 + (m-1)\int_0^1 (1-x)^{m-2}\frac{x^n}{n}\,dx$$

$$= \frac{m-1}{n}\int_0^1 (1-x)^{m-2}x^n\,dx$$

Again integrating by parts, we get

$$= \frac{m-1}{n}\left[\left\{(1-x)^{m-2}\frac{x^{n+1}}{n+1}\right\}_0^1 + (m-2)\int_0^1 (1-x)^{m-3}\frac{x^{n+1}}{n+1}\,dx\right]$$

$$= \frac{(m-1)(m-2)}{n(+1)}\int_0^1 (1-x)^{m-2}x^{n+1}\,dx$$

$$= \frac{(m-1)(m-2)\ldots 2\times 1}{n(n+1)\ldots(n+m-2)}\int_0^1 x^{n+m-2}\,dx$$

$$= \frac{(m-1)(m-2)\ldots 2\times 1}{n(n+1)\ldots(n+m-2)}\left[\frac{x^{n+m-1}}{n+m-1}\right]_0^1$$

$$= \frac{(m-1)(m-2)\ldots 2\times 1}{n(n+1)\ldots(n+m-2)(n+m-1)}$$

Multiplying numerator and denominator by $(n-1)(n-2)\ldots 2\times 1$

$$B(m,n) = \frac{(m-1)(m-2)\ldots 2\times 1 \qquad (n-1)(n-2)\ldots 2\times 1}{n(n+1)\ldots(n+m-2)(n+m-1)(n-1)(n-2)\ldots 2\times 1}$$

$$= \frac{(m-1)!(n-1)!}{1\times 2\ldots(n-2)(n-2)n(n+1)\ldots(n+m-2)(n+m-1)}$$

$$= \frac{(m-1)!(n-1)!}{(n+m-1)!}$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Now if m is only positive integer and n is not, then

$$B(m,n) = \frac{(m-1)!}{n(n+1)\dots(n+m-1)}$$

Theorem

$$\beta(m,n) = 2\int_0^{\pi/2} (\cos\theta\ )^{2m-1} (\sin\theta)^{2n-1}\, d\theta$$

Proof : By definition of beta function

$$\int_0^{\pi/2} sin^{2m-1}\theta\ cos^{2n-1}\theta\ d\theta = \frac{1}{2}\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$$

We know that $\beta\ (m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Let $x = cos^2\theta$ → $dx = 2\cos\theta\ (-\sin\theta\ )d\theta$

$1 - x = sin^2\theta$ $\quad dx = -2\cos\theta\ \sin\theta\ d\theta$

at $x = 0$ $\quad cos^2\theta = 0$ → $\theta = \frac{\pi}{2}$

at $x = 1$ $\quad cos^2\theta = 1$ → $\theta = 0$

$$\beta(m,n) = \int_{\pi/2}^{0} (\ cos^2\theta\ )^{m-1} (\ sin^2\theta\ )^{n-1} (-2\cos\theta\ \sin\theta\ d\theta)$$

$$\beta(m,n) = 2\int_0^{\pi/2} (\cos\theta\ )^{2m-2} (\sin\theta)^{2n-2} \cos\theta\ \sin\theta\ d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} (\cos\theta\ )^{2m-1} (\sin\theta)^{2n-1}\, d\theta$$

Theorem

To show that $\beta\,(m,n) = \beta\,(n,m)$

Proof :

By definition of beta function

$$\beta\,(m,n) = \int_0^1 x^{\mathrm{m}-1}(1-x)^{n-1}\,dx$$

Putting $x = 1-y$ and $dx = -dy$ in above equation, we get

$$\beta\,(m,n) = \int_1^0 (1-y)^{\mathrm{m}-1}\,y^{n-1}\,(-dy)$$

$$\beta\,(m,n) = \int_0^1 y^{n-1}\,(1-y)^{\mathrm{m}-1}\,dy$$

$$\beta\,(m,n) = \beta\,(n,m)$$

Example

Evaluate $\int_0^1 x^4(1-x)^3\,dx$

Solution

$$\int_0^1 x^4(1-x)^3\,dx = \int_0^1 x^{5-1}(1-x)^{4-1}\,dx$$

$$\beta\,(5,4) = \Gamma\,(5)\,\Gamma\,(4)/\Gamma\,(9)$$

$= 4!\;.\;3!\;/\;8!\;\; = 3!/(8.7.6.5) = 1/\;(8.7.5) = 1/280$

Example

Evaluate $I = {}_0\int^1 \left[1 / \sqrt[3]{[x^2 (1-x)]} \right] dx$

Solution

$I = {}_0\int^1 x^{-2/3} (1-x)^{-1/3} dx = {}_0\int^1 x^{1/3-1} (1-x)^{2/3-1} dx$

$= B(1/3,2/3) = \Gamma(1/3)\ \Gamma(2/3) / \Gamma(1)$

Example

$$\frac{\Gamma(6)}{2\Gamma(3)} = \frac{5!}{2\cdot 2!} = \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 2} = 30$$

$$\int_0^\infty x^3 e^{-x} dx = \Gamma(4) = 3! = 6$$

Example **Evaluate** $\int_0^1 x^4 \left(1-\sqrt{x}\right)^5 dx$

Solution: Let $\sqrt{x} = t \implies x = t^2$ so that $dx = 2t\,dt$

$$\int_0^1 x^4 \left(1-\sqrt{x}\right)^5 dx = \int_0^1 (t^2)^4 (1-t)^5 (2t\,dt)$$

$$= 2\int_0^1 t^9 (1-t)^5\, dt$$

$$= 2\, B(10,6)$$

$$= 2\,\frac{\Gamma 10\, \Gamma 6}{\Gamma 16}$$

$$= 2 \times \frac{9!5!}{15!}$$

$$= \frac{2\times1\times2\times3\times4\times5}{15\times14\times13\times12\times11\times10}$$

$$= \frac{1}{11\times13\times7\times15}$$

$$= \frac{1}{15015}$$

$$\therefore \quad \int_0^1 x^4 \left(1-\sqrt{x}\right)^5 dx = \frac{1}{15015}$$